"Ring" – Hamiltonian

$$\begin{bmatrix} E_0 & & & & & & \\ t & E_0 & & & & & \\ 0 & t & E_0 & & & & \\ 0 & 0 & t & E_0 & & & \\ \vdots & \vdots & \vdots & \vdots & \ddots & & \\ 0 & 0 & 0 & 0 & \cdots & E_0 & \\ t & 0 & 0 & 0 & \cdots & t & E_0 \end{bmatrix}$$

basis $\{\varphi_1, \varphi_2, \dots \varphi_N\}$

$\langle \varphi_i | \varphi_j \rangle = \delta_{ij}$

$\langle \varphi_i | H | \varphi_i \rangle = E_0$

$\langle \varphi_i | H | \varphi_{i+1} \rangle = t \quad + h.c$

$\langle \varphi_N | H | \varphi_1 \rangle = t \quad + h.c$

$\langle \varphi_i | H | \varphi_j \rangle = 0$ otherwise

$$\psi_k = \frac{1}{\sqrt{N}} \sum_{j=1}^{N} e^{ikj} \varphi_j \quad \text{with } k = \frac{2\pi}{N} \cdot n \text{ and } n = 1 \dots N$$

$$\left(\text{or } n = -\frac{N}{2}+1, \dots, \frac{N}{2}\right)$$

$$k \neq k' \rightarrow \langle \psi_k | H | \psi_{k'} \rangle = \frac{1}{\sqrt{N}} \left\langle \sum_{j=1}^{N} e^{ikj} \varphi_j \middle| H \middle| \sum_{j'=1}^{N} e^{ik'j'} \varphi_{j'} \right\rangle \frac{1}{\sqrt{N}}$$

$$= \frac{1}{N} \sum_{j=1}^{N} \sum_{j'=1}^{N} e^{-ikj} e^{ik'j'} \langle \varphi_j | H | \varphi_{j'} \rangle$$

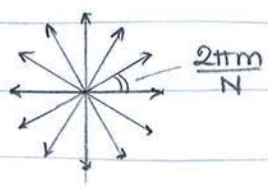
$e^{ik'j} \cdot e^{-ik'j}$

$$= \frac{1}{N} \sum_{j=1}^{N} \sum_{j'=1}^{N} e^{i(k'-k)j} e^{ik'(j'-j)} \langle \varphi_j | H | \varphi_{j'} \rangle$$

$$= \frac{1}{N} \sum_{j=1}^{N} e^{i(k'-k)j} \sum_{j'=1}^{N} e^{ik'(j'-j)} \langle \varphi_j | H | \varphi_{j'} \rangle$$

$$\langle \varphi_j | H | \varphi_{j'} \rangle = f(j'-j) \rightarrow \; = \frac{1}{N} \sum_{j=1}^{N} e^{i(k'-k)j} \sum_{j'=1}^{N} e^{ik'(j'-j)} f(j'-j)$$

$$\sum_{j'=1}^{N} e^{ik'(j'-j)} f(j'-j) = F(k') \rightarrow \; = \frac{1}{N} \sum_{j=1}^{N} e^{i(k'-k)j} F(k') = \frac{F(k')}{N} \cdot \sum_{j=1}^{N} e^{i(k'-k)j}$$



Unit-circle

$$= \frac{F(k')}{N} \cdot \sum_{j=1}^{N} e^{i\frac{2\pi}{N}(n'-n)j} = \frac{F(k')}{N} \cdot \sum_{j=1}^{N} e^{i\frac{2\pi m}{N} \cdot j} = 0$$

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$$k=k' \longrightarrow \langle \psi_k | H | \psi_k \rangle = \frac{1}{N} \sum_{j=1}^{N} \sum_{j'=1}^{N} e^{ik(j'-j)} \langle \varphi_j | H | \varphi_{j'} \rangle$$

$$= \frac{1}{N} \cdot N \cdot \sum_{\Delta j=0}^{N-1} e^{ik\Delta j} \langle \varphi_1 | H | \varphi_{1+\Delta j} \rangle$$

$$= \langle \varphi_1 | H | \varphi_1 \rangle + e^{ik} \langle \varphi_1 | H | \varphi_2 \rangle + e^{ik(N-1)} \langle \varphi_1 | H | \varphi_N \rangle$$

$$= E_0 + e^{ik} \cdot t + e^{ikN} \cdot e^{-ik} \cdot t$$

$$\left(e^{ikN} = e^{i\frac{2\pi n}{N} \cdot N} = e^{i2\pi n} = 1 \right)$$

$$= E_0 + t\left(e^{ik} + e^{-ik}\right)$$

$$= E_0 + 2t\cos(k)$$

$$= E_0 + 2t\cos\left(\frac{2\pi n}{N}\right)$$

$n = 1 \ldots N$ or

$n = -\frac{N}{2}+1, \ldots, \frac{N}{2}$